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LETTER TO THE EDITOR

The thermodynamic limit of the quenched free energy of magnetic systems with random interactions

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Abstract. Assuming translational invariance and random interaction with zero mean value, we show that the thermodynamic limit of the quenched free energy of a disordered magnetic system exists if the annealed free energy also exists. For a Hamiltonian with pair interactions and a class of even distribution functions the limit exists if the interaction potential is square summable.

In a previous letter (Goulart Rosa 1980) it has been shown that the quenched free energy of a magnetic system with random interaction cannot decrease in the regularisation process of replacing part of the coupling constants by their mean values. In systems with probability distribution functions which give zero mean value for the random coupling constants this regularisation mechanism of the disordered system can be thought of as a turning-off process of the interactions between spins and/or the removal of the external (stochastic) magnetic field.

This theorem is now used to discuss the existence of the thermodynamic limit of the quenched free energy of disordered systems. The strategy adopted here is similar to that used in the study of the thermodynamic limit problem of continuum systems. Using the regularisation process we shall show first that the free energy per site of a disordered system, defined on a sequence of boxes tending to infinity, is a monotonically decreasing sequence. The sufficient condition for the convergence of this sequence is obtained demanding the existence of the thermodynamic limit of the annealed free energy of the system, which is well known to be a lower bound for the quenched free energy.

Let us consider a random magnetic system defined by the Hamiltonian

$$H_\Lambda = -\sum_{p\subset\Lambda} J_p\sigma_p(\{\boldsymbol{S}_i\}) \tag{1}$$

where Λ is a finite set of $|\Lambda|$ sites in a d-dimensional lattice; $\sigma_p(\{\boldsymbol{S}_i\})$ describes a general many-body interaction between the spin operators $\boldsymbol{S}_i$, $i\subset p$ and p is a subset of Λ. The influence of external magnetic fields is also contained in equation (1) (Griffiths 1972).

The definition of the system is only completed after assigning to the real random variables $\{J_p\}$ a probability function $\mathscr{P}$. For statistically independent coupling constants we have that

$$\mathscr{P} = \prod_{p\subset\Lambda} P(J_p, \alpha_p) \tag{2}$$

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where α_p is a set of parameters, e.g. the moments which are assumed to be finite, which particularises the distribution. Translational invariance requires that $J_p = J_{p'}$, $\alpha_p = \alpha_{p'}$, if p' is a set obtained from p by translation.

Following Brout (1959) we introduce the quenched and the annealed free energy per site

$$f_\Lambda = \frac{\langle F_\Lambda(\{J_p\}, \{\alpha_p\}, T)\rangle_{\mathcal{P}}}{|\Lambda|} = -\frac{kT}{|\Lambda|}\langle \ln \mathrm{Tr} \exp -\beta H_\Lambda\rangle_{\mathcal{P}} \tag{3}$$

$$g_\Lambda = \frac{\langle G_\Lambda(\{J_p\}, \{\alpha_p\}, T)\rangle_{\mathcal{P}}}{|\Lambda|} = -\frac{kT}{|\Lambda|}\ln \langle \mathrm{Tr} \exp -\beta H_\Lambda\rangle_{\mathcal{P}} \tag{4}$$

where the angular bracket means the configurational average with respect to the probability distribution $\mathcal{P}$; $\beta^{-1} = kT$ is the product of Boltzmann constant k times the absolute temperature T. The existence of the above free energies for finite domains is ensured by the finiteness of the moments $\langle J_p^n\rangle_{\mathcal{P}}$ (Vuillermot 1977).

Theorem. Let us consider a disordered magnetic system defined by equations (1) and (2) with translational invariance and probability distribution functions $P(J_p, \alpha_p)$ such that $\langle J_p\rangle_{\mathcal{P}} = 0$. Then for a sequence of domain Λ tending to infinity in the sense of Van Hove the thermodynamic limit of the quenched free energy exists if the annealed free energy also exists in that limit.

Proof. Without loss of generality let us consider a standard sequence of cubes:

$$\Lambda_n = \{r \in \mathbb{Z}^d; 0 \leq x^i \leq 2^n; i = 1, 2, \ldots, d; n = 1, 2, \ldots\}$$

so that a cube Λ_n with edge $L_n = 2^n$ may be thought of as composed by the cubes $\Lambda_{n-1}^{(j)}$, $j = 1, 2, \ldots, 2^d$ with edge $L_{n-1} = 2^{n-1}$. By turning off all the coupling constants between spins which belong to distinct cubes $\Lambda_{n-1}^{(j)}$ of the decomposition of Λ_n we obtain the following inequality for the quenched free energies:

$$\langle F_{\Lambda_n}(\{J_p\}, \{\alpha_p\}, T)\rangle_{\mathcal{P}} \leq \sum_{j=1}^{2^d} \langle F_{\Lambda_{n-1}^{(j)}}(\{J_{p_j}\}, \{\alpha_{p_j}\}, T)\rangle_{\mathcal{P}}. \tag{5}$$

Using the translational invariance hypothesis

$$\langle F_{\Lambda_{n-1}^{(j)}}(\{J_{p_j}\}, \{\alpha_{p_j}\}, T)\rangle_{\mathcal{P}} = \langle F_{\Lambda_{n-1}}(\{J_p\}, \{\alpha_p\}, T)\rangle_{\mathcal{P}} \tag{6}$$

and the relationship between the volumes $|\Lambda_n| = 2^d|\Lambda_{n-1}|$ we obtained from equation (5) that the sequence f_{Λ_n} is monotonically decreasing in n i.e.

$$f_{\Lambda_n} \leq f_{\Lambda_{n-1}} \qquad \forall n. \tag{7}$$

As a consequence of Jensen's inequality we know that

$$g_\Lambda \leq f_\Lambda. \tag{8}$$

Therefore if g_Λ exists in the limit $|\Lambda| \to \infty$ the sequence f_{Λ_n} is also convergent, proving the theorem. It is worthwhile remarking that g_Λ is a universal lower bound in the sense that equation (8) is true for any Hamiltonian and distribution function. However its main advantage as a lower bound stems from the fact that averaging the trace

$$\langle Z_\Lambda\rangle_{\mathcal{P}} = \langle \mathrm{Tr} \exp -\beta H_\Lambda\rangle_{\mathcal{P}} \equiv \mathrm{Tr} \exp -\beta H_\Lambda^{\mathrm{eff}}(\{J_p\}, \{\alpha_p\}, T) \tag{9}$$

will generate an effective non-random (regular) Hamiltonian H^{eff}, thus allowing us to apply all the known results for existence of the thermodynamic limit of regular magnetic systems (Ruelle 1969) in the realm of disordered systems.

As an application of this result let us consider the class of random systems defined by the Hamiltonian

$$H_\Lambda = -\sum_{\substack{i<j\\ i\neq j}} \varepsilon_{ij}\Phi_{ij}s_is_j \qquad s_i = \pm 1 \tag{10}$$

where for convenience we have separated the non-random part of the interaction, Φ_{ij}, from the random term, which is distributed by the distribution function

$$\mathscr{G} = P(\varepsilon_{ij}, \langle\varepsilon_{ij}^2\rangle) = (\tfrac{1}{2}\pi\langle\varepsilon_{ij}^2\rangle)^{1/2}\exp[-\varepsilon_{ij}^2/2\langle\varepsilon_{ij}^2\rangle] \tag{11}$$

where $\langle\varepsilon_{ij}^2\rangle$ are uniformly bounded, i.e. $a \leq \langle\varepsilon_{ij}^2\rangle \leq b$. The evaluation of the annealed free energy of this system is straightforward:

$$g_\Lambda = -kT\ln 2 - \frac{\beta}{4|\Lambda|}\sum_{\substack{i<j\\ i\neq j}}\langle\varepsilon_{ij}^2\rangle\Phi_{ij}^2. \tag{12}$$

Here we open parentheses to show that the divergence of g_Λ at $T=0$ does not jeopardise our argumentation because this situation can be easily remedied. This is accomplished noting that the annealed free energy is negative and it is a concave function of T with a maximum at $T_{\max}$, yielding a negative annealed entropy $\partial g/\partial T = -S_{\text{ann}}$ for all temperatures below $T_{\max}$. According to Brout the physical thermodynamic description of the disordered system is derived from the quenched free energy and its entropy must be positive at all temperatures for Ising spins systems (Toulouse and Vannimenus 1980). Therefore an improved lower bound depurated of unphysical features can be constructed from equation (12):

$$f_\Lambda \geq g_\Lambda^* \tag{8a}$$

$$g_\Lambda^* = \begin{cases} g_\Lambda(T) & T > T_{\max} \\ g_\Lambda(T_{\max}) & T \leq T_{\max}. \end{cases} \tag{12a}$$

Therefore the infinite volume limit of annealed free energy exists if

$$\lim_{|\Lambda|\uparrow\infty}\frac{1}{|\Lambda|}\sum_{\substack{i<j\\ i,j\subset\Lambda}}\Phi_{ij}^2 < \infty \tag{13}$$

where we have used the supremum of the set $\{\langle\varepsilon_{ij}^2\rangle\}$ to simplify condition (13). The translational invariance of the potential $\Phi_{ij} = \Phi(|i-j|)$ can be used to carry out the summation over one of the site indices, further simplifying this expression. The above sufficient condition is also true for a class of even probability function, $\mathscr{P}$ with the moments of the random variables satisfying the relationship

$$\langle\varepsilon_{ij}^{2n}\rangle_{\mathscr{P}} \leq (2n-1)!!\langle\varepsilon_{ij}^2\rangle_{\mathscr{G}}^n \tag{14}$$

where $\langle\varepsilon_{ij}^2\rangle_{\mathscr{G}}$ is the variance in equation (11). This can be easily verified imposing that

$$\langle Z_\Lambda(\{\Phi_{ij}\}, \{\varepsilon_{ij}\}, T)\rangle_{\mathscr{P}} \leq \langle Z_\Lambda(\{\Phi_{ij}\}, \{\varepsilon_{ij}\}, T)\rangle_{\mathscr{G}} \tag{15}$$

and performing a high-temperature expansion for the partition functions. We note that the symmetric double-peaked distribution function (ε_{ij} takes only the values $\pm\varepsilon_0$ with probability one half), which is together with the Gaussian function one of the most commonly used distribution functions in the theory of spin glasses, is included in the class of probability functions defined by equation (14). A similar class of distribution functions has been considered by Khanin and Sinai (1979) in their study of the thermodynamic limit of the particular system with potential $\Phi(|i-j|)=|i-j|^{-\alpha d}$ where d is the dimensionality of the lattice. They proved that the thermodynamic limit of the quenched free energy exists if $\alpha > \frac{1}{2}$ which coincides with the general condition (13).

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